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$$ay + bxy + cy^2 + t_{n>2} = 0, \quad \ldots\ldots\ldots\ldots \quad (g)$$

which does not contain x^2. (Anal. Geom. art. 425). Compute the Hes. of (g),

$$\left(\frac{du}{dx}\right) = by + t_{n>1}, \quad \left(\frac{du}{dy}\right) = a + bx + 2cy + t_{n>1}, \quad \left(\frac{d^2u}{dx^2}\right) = t_{n>0},$$

$$\left(\frac{d^2u}{dxdy}\right) = 2c + t_{n>0}, \qquad \left(\frac{d^2u}{dy^2}\right) = b + t_{n>0}.$$

$\therefore H = t_{n>0} = 0$; and since there is no constant term the Hessian passes through the point of inflection as before stated.

This method may be applied in a similar manner to the singularities of higher orders which, as has been shown by Cayley, are merely the superposition or coincidence of several of these simple singularities.

BIPOLAR EQUATIONS - CARTESIAN OVALS.

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Every system of co-ordinates is specially adapted to the expression of some particular properties of those curves which admit of simple equations in that system. The object of this paper is to discuss the Bipolar system of point-coordinates and particularly the Cartesian Ovals, (the loci of Bipolar equations of the first degree).

Let the fixed points A_1 and A_2 be taken as poles, and denote by a the distance between the poles; then ρ_1 and ρ_2, denoting the distances of any point from the poles, are the bipolar coordinates of that point. We observe that any two points symmetrically situated with respect to the line A_1A_2 have the same pair of coordinates, hence the locus of every equation in bipolar coordinates is symmetrical with respect to the axis A_1A_2.

The equations of transformation to rectangular and to polar coordinates, taking A_1 for origin or pole and A_1A_2 for axis of x or initial line, are

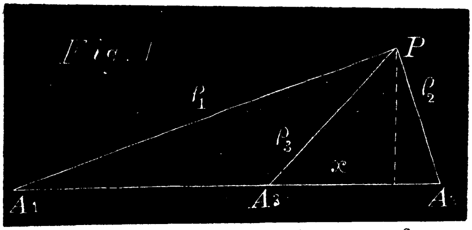
$$\left.\begin{aligned} \rho_1^2 &= x^2 + y^2 = \rho^2 \\ \rho_2^2 &= x^2 + y^2 - 2ax + a^2 = \rho^2 - 2a\rho\cos\theta + a^2 \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

The values of ρ_1 and ρ_2 in terms of x and y being square roots, we observe that the rational rectangular equation obtained by transformation from any bipolar equation will always include the several loci which result from giving to ρ_1 and ρ_2 the ambiguous sign. Thus $\pm\rho_1 \pm \rho_2 = na \ldots (2)$ will be included in a single rectangular equation. We shall therefore regard the results of different selections of the signs as the equations of

different branches of the same curve. These branches are not all possible at once, thus in (2) if $n>1$, $\rho_1+\rho_2=na$ is the only possible branch, (since for every point we must have $\rho_1 \sim \rho_2<a$) and represents an ellipse; while if $n<1$, this branch is impossible and we have $\rho_1-\rho_2=\pm na$ representing the two branches of an hyperbola. Thus (2) is the general equation of a conic with foci at the poles.

Bipolar equations may sometimes be rendered homogeneous by the introduction of a third coordinate, ρ_3, representing the distance of the point from a third pole A_3, taken on the axis A_1A_2; thus forming a redundant system of coordinates analogous to the trilinear system. To deduce the relation which must always exist between ρ_1, ρ_2, and ρ_3, denote in Fig. 1 the distance between A_1 and A_2 by a_3, between A_2 and A_3 by a_1, between A_3 and A_1 by a_2; for the sake of symmetry so taking the signs of these quantities that

$$a_1+a_2+a_3=0. \qquad (3)$$



(In the figure a_3 is regarded as negative.) Denote the projection of ρ_3 upon the axis by x; then

$$\rho_1^2-\rho_2^2=(a_2+x)^2-x_2$$
$$=a_2^2+2a_2x$$
$$\rho_2^2-\rho_3^2=(a_1-x)^2-x_2$$
$$=a_1^2-2a_1x.$$

Eliminating x, $a_1\rho_1^2-a_1\rho_3^2+a_2\rho_2^2-a_2\rho_3^2=a_1a_2^2+a_2a_1^2$, or since $a_1+a_2=-a_3$

$$a_1\rho_1^2+a_2\rho_2^2+a_3\rho_3^2+a_1a_2a_3=0. \qquad (4)$$

By means of this relation we may eliminate the absolute term from a bipolar equation. We may use it also to eliminate one of the variables, for instance ρ_2, from a bipolar equation thus obtaining the bipolar equation of the same locus referred to the poles A_1 and A_3.

The general equation of the first degree may be written in the form

$$\pm m\rho_1 \pm l\rho_2 \pm na=0, \qquad (5)$$

in which we regard each of the symbols as denoting a positive quantity. The locus of (5) consists of two branches, (except when the equation becomes that of an ellipse,) and two only. For in the first place it is obvious that the terms can not all have the same sign, we may therefore write two of the terms with the positive and one with the negative sign; now since ρ_1, ρ_2 and a form the sides of a triangle, of which one side can not exceed the sum of the other two, it is impossible that the negative term, which must in absolute value equal the sum of the other two, should have the least of the three coefficients, m, l, and n. We therefore obtain the equations of the real branches by giving to *one or the other of the two greater coefficients the negativ sign.*

Infinite values of ρ_1 and ρ_2, it is evident, can never satisfy the equation unless the coefficients m and l are equal and have opposite signs, in which case the curve becomes an hyperbola. In all other cases therefore the locus consists of two closed branches which are called Cartesian Ovals.

One pole at least is within both ovals. For since either the sum or the difference of $m\rho_1$ and $l\rho_2$ is constant we have for any two points P and P' of the same branch, $m(\rho_1 \sim \rho'_1) = l(\rho_2 \sim \rho'_2)$.
Now let P and P' be the points in which the oval cuts the axis. If the pole A_1 is outside of the oval, the difference $\rho_1 - \rho'_1$ becomes the axis of the oval, if it is within it becomes the difference between two segments of the axis of the oval, and will be the less the nearer the pole is to the middle point of this axis. Now if $m > l$, the difference $\rho_1 \sim \rho'_1$ will be less than $\rho_2 \sim \rho'_2$; hence A_1 will be within the oval, and will be nearer than A_2 is to the middle point, in case the latter also is within the oval. That is, *corresponding to the greater of the coefficients m and l, we have a pole which is within each oval.*

Supposing $m > l$, if $n > l$ the equations of the branches will be

$$\left.\begin{aligned} m\rho_1 + l\rho_2 - na &= 0, \\ -m\rho_1 + l\rho_2 + na &= 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (6)$$

In the first $m\rho_1$ is always less than the constant na, in the second it is always greater than na; therefore since A_1 from which ρ_1 is measured is within both branches, the first branch lies entirely within the other. If however $m = n$, the branches will meet in the single point A_2 whose coordinates $(a, 0)$ satisfy both equations.

If $m > l > n$, the equations will be

$$\left.\begin{aligned} m\rho_1 - l\rho_2 + na &= 0, \\ -m\rho_1 + l\rho_2 + na &= 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (7)$$

In this case *the pole A_2 will be outside of each oval.* For, at any point between A_1 and A_2 we have $\rho_1 + \rho_2 = a$, combining this with equations (7) we have

$$\rho_1 = \frac{l \mp n}{m + l} a. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

Each of these values is positive and less than a; therefore the branches cut A_1A_2 between the poles, and A_1 being within, A_2 must be outside of each oval. The greater value of ρ_1 in (8) is derived from the second equation in (7), therefore that equation belongs to the outer branch. In both the cases (7) and (8), *the outer branch is that in which the greater coefficient has the negative sign.*

We will now show that the Cartesian

$$m\rho_1 \pm l\rho_2 \pm na_3 = 0 \quad \ldots\ldots\ldots\ldots \quad (9)$$

admits of a homogeneous tripolar equation of the first degree. Transposing and squaring (9) $m^2\rho_1{}^2 \pm 2ml\rho_1\rho_2 + l^2\rho_2{}^2 = n^2a_3{}^2$.
This equation is rendered homogeneous by multiplying by

$$1 = -\frac{\rho_1{}^2}{a_2a_3} - \frac{\rho_2{}^2}{a_3a_1} - \frac{\rho_3{}^2}{a_1a_2}$$

derived from (4), which gives

$$\left(m^2 + n^2\frac{a_3}{a_2}\right)\rho_1{}^2 \pm 2ml\rho_1\rho_2 + \left(l^2 + n^2\frac{a_3}{a_1}\right)\rho_2{}^2 = -n^2\frac{a_3{}^2}{a_1a_2}\rho_3{}^2 \ldots (10)$$

This equation will reduce to one of the first degree when the first member is a square, that is, when

$$\left(m^2 + n^2\frac{a_3}{a_2}\right)\left({}^2 + n^2\frac{a_3}{a_1}\right) = m^2l^2,$$

or

$$l^2a_1 + m^2a_2 + n^2a_3 = 0. \quad \ldots\ldots\ldots\ldots \quad (11)$$

The values of a_1 and a_2 must also satisfy (3), $a_1 + a_2 + a_3 = 0$,

hence we derive $a_1 = \dfrac{n^2 - m^2}{m^2 - l^2}a_3$ and $a_2 = \dfrac{l^2 - n^2}{m^2 - l^2}a_3. \quad \ldots\ldots \quad (12)$

To reduce equation (10) most symmetrically, multiply by a_1a_2,

$$a_1(m^2a_2 + n^2a_3) \pm 2mla_1a_2\rho_1\rho_2 + a_2(l^2a_1 + n^2a_3)\rho_2{}^2 = -n^2a_3{}^2\rho_3{}^2,$$

substituting from (11), $-l^2a_1{}^2\rho_1{}^2 \pm 2mla_1a_2\rho_1\rho_2 - m^2a_2{}^2\rho_2{}^2 = -n^2a_3{}^2\rho_3{}^2$.

Hence

$$la_1\rho_1 \mp ma_2\rho_2 \pm na_3\rho_3 = 0. \quad \ldots\ldots\ldots\ldots \quad (13)$$

The three poles in this equation constitute three foci of the Cartesian, with respect to any pair of which the curve has similar properties. For, eliminating ρ_2 between (13) and (9) [observing that in the first ambiguous signs of these equations, the upper signs correspond], we have

$$(l^2a_1 + m^2a_2)\rho_1 \pm mna_2a_3 \pm nla_3\rho_3 = 0,$$

or by (11)

$$-n^2a_3\rho_1 \pm mna_2a_3 \pm nla_3\rho_3 = 0$$

or

$$\pm n\rho_1 \pm ma_2 \pm l\rho_3 = 0.$$

The three bipolar and the tripolar equation are therefore as follows—

$$\left.\begin{aligned} \pm m\rho_1 \pm l\rho_2 \pm na_3 &= 0 \\ \pm n\rho_1 \pm ma_2 \pm l\rho_3 &= 0 \\ \pm la_1 \pm n\rho_2 \pm m\rho_3 &= 0 \\ \pm la_1\rho_1 \pm ma_2\rho_2 \pm na_3\rho_3 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (14)$$

Since the three bipolar equations have the same coefficients their order only being changed, we see that the distinction between the cases (6) and (7) is not a distinction between varieties of the curve, but is due solely to the relative position of the foci which are taken as poles. Let the order of magnitude of the coefficients be $n > m > l$; the first equation shows that A_1 is within the ovals and nearer to the middle than A_2, while the third equation, in which the coefficient of the absolute term is the least, (as in eq. (7),)

shows that A_2 is within and A_3 outside of the ovals. We may call A_1, in this case, the first focus, A_2 the second, and A_3 the third focus.

When $m = l$, the Cartesian $m\rho_1 \pm l\rho_2 \pm na_3 = 0$ becomes a conic, and (12) shows that the pole A_3 is at infinity. The properties expressed in the second and third bipolar equations of (14) then become the property of a focus and directrix. This may be proved as follows—Writing the equation in the form

$$ee'\rho_1 \pm e\rho_2 \pm a_3 = 0$$

the second equation of (14) becomes

$$\rho_1 \pm ee'a \pm e\rho_3 = 0 \text{ or } \qquad \rho_1 = \pm e(e'a_2 \pm \rho_3).$$

When $e > 1$ and e' near to 1, the two branches take the forms

$$\rho_1 = \pm e(e'a_2 - \rho_3). \quad \ldots \ldots \ldots \quad (15)$$

If $e < 1$ and e' near to 1, the inner branch will take one of the forms in (15). Now $e'a_2 - \rho_3$ expresses the distance of a point from the circumference of a circle whose centre is A_3, and (15) expresses that the distance of any point of the curve from A_1 is in the fixed ratio e to its distance from this circumference. When $e' = 1$, this circumference becomes a straight line and e is the excentricity of the conic, which consists of two branches or one according as $e > 1$ or $e < 1$.

Transforming to polar coordinates by (1) after squaring the equation

$$m\rho_1 \pm na_3 = \mp l\rho_2$$

we obtain

$$\rho^2 + \frac{2l^2a_3}{m^2 - l^2}\rho\cos\theta \pm \frac{2mna_3}{m^2 - l^2}\rho + \frac{n^2 - l^2}{m^2 - l^2}a_3^{\,2} = 0, \quad \ldots \quad (16)$$

the polar equation of the Cartesian referred to one of its foci and the axis.

If $n = l$, (16) reduces to $\rho = 0$ and

$$\rho + \frac{2l^2a_3}{m^2 - l^2}\cos\theta \pm \frac{2mla_3}{m^2 - l^2} = 0. \quad \ldots \ldots \ldots \quad (17)$$

This last is the equation of the Limaçon of Pascal or curve generated by increasing and diminishing the radius vector of a circle (pole in the circumference) by the same constant. The bipolar equation

$$m\rho_1 \pm l\rho_2 \pm la_3 = 0 \quad \ldots \ldots \ldots \quad (18)$$

therefore represents the limaçon accompanied by its node. Making $n = l$ in (12) we see that A_3 coincides with the node at A_1. If $m > l$, the first and second foci coincide at an acnode or conjugate point, A_1 $(0, a_3)$ being the only point which satisfies the equation $m\rho_1 \pm l(\rho_2 - a_3) = 0$ while $-m\rho_1 + l\rho_2 + la_3 = 0$ represents the continuous curve. If $m < l$, A_1 satisfies both branches, the second and third foci coinciding at a crunode or double point.

If $n = 0$, the first bipolar equation becomes $m\rho_1 - l\rho_2 = 0 \quad \ldots (19)$ and the distances of the third pole from A_1 and A_2 are, making $n=0$ in (12),

$$a_2 = \frac{l^2}{m^2 - l^2} a_3 \qquad \text{and } a_1 = \frac{-m^2}{m^2 - l^2} a_3 \ldots\ldots \quad (20)$$

Both the second and third bipolar equations of (14) now reduce to

$$\rho_3 = \pm \frac{ml}{m^2 - l^2} a_3.$$

Eq. (19) therefore represents a circle the third pole now coinciding with the centre, and the radius being a mean proportional between the distances (20) of the poles from the centre.

This point we shall designate by C; a_1 and a_2 being of opposite signs, C cannot be between the poles A_1 and A_2; for in $a_1 + a_2 + a_3 = 0$ the distance between the extreme points differs in sign from each of the other distances. We may therefore denote the absolute values of CA_1 and CA_2 by l^2c and m^2c, $A_1A_2 = (m^2 - l^2)c$, and from (12) we see that for any Cartesian the distances a_1 a_2 and a_3 are $n^2c - m^2c$, $m^2c - l^2c$ and $l^2c - n^2c$, thus it appears that C is a definite point for a given Cartesian being on the same side of all three poles and at distances from them proportional to l^2, m^2 and n^2; the coefficients being associated with the poles as they are in the tripolar equation of (14). The first focus is therefore the nearest to, and the third farthest from C.

C is the centre of the primitive circle of the limaçon (17), for using the notation just introduced the equation of that circle is

$$\rho + 2l^2c \cos\theta = 0$$

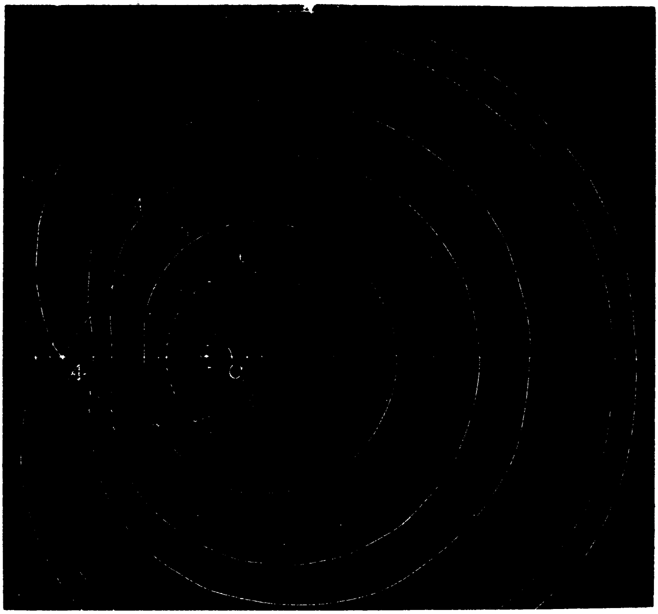
the distance of its centre from A_1 is therefore $-l^2c$, but this is the value of the distance CA_1 and the negative sign shows that it is measured in the direction toward C since we have taken the direction from C to either of the poles as positive.

The Cartesians $\qquad 2\rho_1 \pm \rho_2 \pm na = 0$

may be constructed by a method analogous to that for constructing an ellipse or hyperbola with a thread (The thread passing from the pencil, to which it is attached by means of a hole drilled in the lead, around the needle at A_1 and back to the pencil). The curves in the accompanying diagram have been constructed in this manner for various values of n. The poles A_1 and A_2 and the values of l and m, being common to the whole series the point C is also common. In No. 1 $n > 2$ the third pole is on the left of A_2 and is the third focus. When $n = 2$, the third pole coincides with A_2 and we have No. 2, the crunodal limaçon. (This curve does not generally pass through C). In No. 3 $2 > n > 1$, the third pole has passed to the right of A_2 and is the second focus. When $n = 1$, it coincides with A_1 and we have No. 4 the acnodal limaçon, the inner branch having contracted to the conjugate point A_1. In No. 5, $n < 1$, the third pole has passed to the right of

A_1 and is the first focus. Finally when $n=0$, both branches coincide with the circle No. 6 and the third pole coincides with C.

It will be observed that the conic and the limaçon fulfil the same condition as to the relatve values of the distances from C to the three poles, but in the case of the conic the absolute values of these distances become infinite. Accordingly we see in the inner branches of 3 and 5 an approximation to the elliptic form, and in 1 we have an approximation to the form of an hyperbola in the neighborhood of the second and third foci.



If s denote the arc of the Cartesian and ψ_1 denote the angle the curve makes with ρ_1, evidently $\cos\psi_1 = \frac{d\rho_1}{ds}$ &c.

Differentiating the bipolar equation $m d\rho \pm l d\rho_2 = 0$, whence

$$m \cos\psi \pm l \cos\psi_2 = 0.$$

Thus the direction of the curve makes angles with the focal distances whose cosines are inversely as the coefficients in the bipolar equation or directly as the coefficients in the tripolar equation. When applied to the focus at infinity in the case of the conic we have the property that the cosine of the angle between the tangent and focal radius vector equals the product of the eccentricity by the cosine of the angle between the tangent and axis.

In the polar equation (16) a single value of θ corresponds to four values of ρ, denoting these by ρ' and ρ'' corresponding to the upper, and ρ''' and ρ'''' to the lower sign we have

$$\rho'\rho'' = \rho'''\rho'''' = \frac{n^2 - l^2}{m^2 - l^2} a_3{}^2 \quad \ldots\ldots\ldots\ldots (21)$$

$$\rho' + \rho'' = \frac{-2a}{m^2 - l^2}\left(l^2\cos\theta + mn\right),\ \rho''' + \rho'''' = \frac{-2a}{m^2 - l^2}\left(l^2\cos\theta - mn\right) \ldots (22)$$

Now since $\rho'\rho'' = \rho'''\rho''''$ it is evident that for a pole within the ovals ρ' and ρ'' must belong to different branches while for a pole outside of the ovals they must belong to the same branch. Since $\rho'\rho''$ is constant the inverse of the curve described by ρ' is similar to that described by ρ''. Thus the inverse of a Cartesian with respect to either focus is a similar curve, the inner inverting into an outer branch if the focus is the first or second, but into an inner branch if it is the third focus. Thus the inverse of the acnodal limaçon with respect to the double focus is an ellipse similar to that which vanishes at that point, but its inverse with respect to the single focus is similar to itself. The inverse of the crunodal limaçon with respect to the double point which is at once on each branch consists of the two branches of an hyperbola with asymptotes parallel to the tangents at the double point. From (21) we also see that for the third focus the product of a secant and its external segment is constant, and the four tangents to the curve from this point are equal.

From (22) we have $\rho' + \rho'' - \rho''' - \rho'''' =$ a constant. . . . . . (23) Now when A_1 is the first focus l is the least of the three coefficients and by (21) ρ' and ρ'' have the same sign, while the pole being within the ovals ρ''' and ρ'''' are of the opposite sign. Let ρ''' belong to the same branch as ρ', then $\rho' - \rho'''$ and $\rho'' - \rho''''$ will denote the parts of a straight line passing through A_1 intercepted by the ovals and will have the same sign. Therefore (23) expresses that the *sum* of the intercepts of the ovals upon a line passing through the *first* focus is constant. If A_1 is the second focus, l is the middle coefficient and ρ' and ρ'' have opposite signs. Therefore (23) expresses that the *difference* of the intercepts upon a line passing through the *second* focus is constant. If A_1 is the third focus $\rho' - \rho'''$ and $\rho'' - \rho''''$ will be the intercepts between the branches, and will be of opposite signs since ρ' and ρ'' now belong to the same branch; therefore (21) will express that for a line passing through the *third* focus the *difference* of the intercepts *between* the two branches is constant.

The bipolar equation of the Cartesian takes a simple form also when referred to a focus and the point C as poles. To transform to A_1 and any other point of the axis, combine the equation with the general relation (4) so as to eliminate ρ_2. Thus, squaring, we have

$$m^2\rho_1^{\,2} \pm 2mna_3\rho_1 + n^2a_3^{\,2} = l^2\rho_2^{\,2};$$

multiplying by a_2 and substituting from (4),

$$(m^2a_2 + l^2a_1)\rho_1^{\,2} \pm 2mna_2a_3\rho_1 + l^2a_3\rho_3^{\,2} + n^2a_2a_3^{\,2} + l^2a_1a_2a_3 = 0.$$

This is referred to the focus A_1 and any other point A_3, giving to a_2 and a_1 the values in (20) A_3 coincides with C. Since a_2 is here the distance between the poles, we substitute from (20)

$$a_3 = \frac{m^2 - l^2}{l^2} a_2 \qquad \text{and} \qquad a_1 = -\frac{m^2}{l^2} a_2$$

giving $\rho_3^2 \pm \frac{2mn}{l^2} a_2 \rho_1 + \left[\frac{m^2n^2}{l^4} - \left(\frac{m^2}{l^2} + \frac{n^2}{l^2}\right)\right] a_2^2 = 0 \; . \; . \; . \; . \; . \; (24)$

a relation involving the square of one coordinate with the first power of the other.

If in this equation we suppose $n = 0$ or $m = 0$ we obtain the equation of a circle as before. If $n = l$, (24) becomes

$$\rho_3^2 \pm 2\frac{m}{l} a_2 \rho_1 - a_2^2 = 0, \quad \ldots \ldots \ldots \ldots (25)$$

this is therefore the bipolar equation of the limaçon referred to its node and the centre of the primitive circle. If $m < l$ we have the crunodal limaçon; if $m > l$, the lower sign gives the acnodal limaçon, the upper sign giving an equation satisfied only by A_1, the point $(0, a_2)$. If in (25) we make $m = l$ we have $\rho_3^2 - 2a_2\rho_1 - a_2^2 = 0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (26)$ the bipolar equation of the intermediate case, the Cardioid, which is therefore a Cartesian whose three foci coincide at the cusp. Since in (26) ρ_3 is the distance of a point from the centre of the primitive circle, $\rho_3^2 - a_2^2$ is the square of the tangent upon this circle; and (26) expresses that the square of the tangent from any point of the Cardioid to the primitive circle is a mean proportional to the diameter and the distance of the point from the cusp.

We may now find the polar equation of the Cartesian referred to C by transformation from (24), making, see (1), $\rho_3 = \rho$ and $\rho_1^2 = \rho^2 - 2a_2\rho\cos\theta + a_2^2$. For the sake of abridgement we put for the present

$$\frac{mn}{l^2} = p \qquad \text{and} \qquad \frac{m^2}{l^2} + \frac{n^2}{l^2} = s,$$

(22) then becomes $\rho_3^2 \pm 2pa_2\rho_1 + (p^2 - s)a_2^2 = 0$.

Transforming by the above

$$[\rho^2 + (p^2 - s)a_2^2]^2 = 4p^2a_2^2(\rho^2 - 2a_2\rho\cos\theta + a_2^2)$$

expanding $\rho^4 - 2(s + p^2)a_2^2\rho^2 + 8p^2a_2^3\rho\cos\theta + [(p^2 - s)^2 - 4p^2]a_2^4 = 0$.

By adding and subtracting $(s+p^2)^2a_2^4$ we may write this equation in the form

$$[\rho^2 - (s + p^2)a_2^2]^2 + 8p^2a_2^3\rho\cos\theta - 4p^2(s + 1)a_2^4 = 0.$$

In rectangular coordinates

$$[x^2 + y^2 - (s + p^2)a_2^2]^2 + 8p^2a_2^3[x - \tfrac{1}{2}(s + 1)a_2] = 0.$$

In this equation $a_2 = l^2c$ the value of the distance CA_1; substituting and restoring the values of p and s,

$$[x^2 + y^2 - (l^2m^2 + m^2n^2 + n^2l^2)c^2]^2 + 8l^2m^2n^2c^3[x - \tfrac{1}{2}(m^2 + n^2 + l^2c)] = 0 \ldots (27)$$

an equation of the form $S^2 + k^3L = 0$ where $S = 0$ is the equation of a circle whose centre is at C, and $L = 0$ is the equation of a straight line perpendicular

to the axis. If we denote the distances CA_1, CA_2, CA_3, by a_1, a_2 and a_3 we may substitute in (27) $l^2c = a_1$ $m^2c = a_2$ $n^2c = a_3$ which gives

$$[x^2+y^2-(a_1a_2+a_2a_3+a_3a_1)]^2+8a_1a_2a_3[x-\tfrac{1}{2}(a_1+a_2+a_3)]=0\ldots.(28)$$

If one of the constants as $a_1 = 0$, the equation reduces to $S^2 = 0$, the equation of a pair of circles coincident with $S = 0$. The form of the equation $S^2 + k^3L = 0$ shows that the Cartesian touches the line $L = 0$ in the two points where this line cuts the circle $S = 0$. Hence $L = 0$ is the double tangent of the outer branch.

If we now transform the equation to any other rectangular axes it will take the form $S'^2 + k^3L' = 0$ in which $S' = 0$ is the transformed equation of $S = 0$ and $L' = 0$, that of $L = 0$; therefore the general equation of the Cartesian in rectangular coordinates is of the form $S^2 + k^3L = 0$, the centre of the circle S determining the point C, while the axis is perpendicular to the line $L = 0$. Suppose the equation when transformed to the origin C and the axis as axis of x, to be

$$(x^2 + y^2 - a^2)^2 + k^3(x - p) = 0, \ldots\ldots\ldots (29)$$

then comparing eq. (28) we see that a_1 a_2 and a_3 are the roots of the cubic

$$a^3 - 2pa^2 + a^2a - \tfrac{1}{8}k^3 = 0.\ \ldots\ldots\ldots\ldots (30)$$

If in (29), $a < p$, the line $x = p$ actually touches the outer branch which has a reentrant portion like Nos. 1, 2 and 3 in the diagram. If $a = p$ which is the case in No. 4, (in which $a_1 = 1$, $a_2 = 4$ and $a_3 = 1$), $x = p$ has contact of the third order with the curve. If $a < p$, as in No. 5, the double tangent does not really touch the curve.

It is noticeable that with given values of a and p in (29) the Cartesian cannot be made to approach as near as we choose to the coincident circles $S^2 = 0$ when these are cut by $L = 0$, for the whole Cartesian is always on one side of $L = 0$; yet $k = 0$ reduces (29) to $S^2 = 0$. In fact in this case, since $p < a$, the cubic (30) will be found to have but one real root when $k = 0$. Nevertheless taking $a_3 = 0$, real values of a_1 and a_2 can be found. For comparing (28) and (29) we see we are no longer bound to satisfy the condition $p = \frac{1}{2}(a_1 + a_2 + a_3)$ when $k = 0$, hence we have only to fulfil the condition $a_1a_2 + a_2a_3 + a_3a_1 = 0$ which reduces to $a_1a_2 = 0$. Thus the foci A_1 and A_2 are real and subject only to the condition that a the radius shall be a mean proportional between them.

Since in (29) the expression $x^2 + y^2 - a^2$ is the constant product of the segments of a chord or secant to the circle through (x, y) and $x - p$ is the distance to the line $x = p$, eq. (29) expresses that the distance from any point of the Cartesian to the straight line $L = 0$ is proportional to the square of the product of the segments of a chord or to the fourth power of the tangent to the circle $S = 0$.